Discrete Math
4/28

Quick Review of Logs

Logarithm

文A Language ☆ Watch ✎ Edit

In mathematics, the **logarithm** is the inverse function to exponentiation. That means the logarithm of a given number x is the exponent to which another fixed number, the *base* b, must be raised, to produce that number x. In the simplest case, the logarithm counts the number of occurrences of the same factor in repeated multiplication; e.g., since $1000 = 10 \times 10 \times 10 = 10^3$, the "logarithm base 10" of 1000 is 3, or $\log_{10}(1000) = 3$. The logarithm of x to *base* b is denoted as $\log_b(x)$, or without parentheses, $\log_b x$, or even without the explicit base, $\log x$, when no confusion is possible, or when the base does not matter such as in big O notation.

Quick Review of Logs

- $\log_b(x) = w \equiv b^w = x$
  - which implies that $b^{\log_b(x)} = x$
- $\log_x(y)$ is approximately the number of digits needed to represent y in base-x
  - The exact number is $1 + \lfloor \log_x(y) \rfloor$
- $\log_b(xy) = \log_b(x) + \log_b(y)$ and its related identities:
  - $\log_a(b^n) = n \log_a(b)$ and
  - $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
- $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$ – i.e., changing base is multiplying by a constant

How do we show that $\log_2 3$ is irrational?

$\log_2 3$ is irrational (Question 47)

Proof. $\log_2 3 \notin Q$

We proceed by contradiction.

Assume that $\log_2 3 \in Q$. Reminder that $Q = \{ x/y \mid x \in Z \text{ and } y \in Z^+\}$

So, $\log_2 3 = x/y$ where $x \in Z$ and $y \in Z^+$

Moving the denominator over, $y * \log_2 3 = x$

Using the log power rule, $\log_2 (3^y) = x$

By definition of logs, $3^y = 2^x$

Considering that both x and y are integers, 3^y and 2^x must also be integers. The fundamental theorem of arithmetic states that any integer can be represented as a unique product of prime numbers. The two exponentiations clearly don't share the same prime factors so walla! A clear contradiction. Because the assumption led to a contradiction, it must be false and $\log_2 3 \notin Q$.

How do we show that $\log_8 9$ is irrational?

$\log_8 9$ is irrational

Proof. $\log_8 9 \notin Q$

We proceed by contradiction.

Assume that $\log_8 9 \in Q$. Reminder that $Q = \{ x/y \mid x \in Z \text{ and } y \in Z^+\}$

So, $\log_8 9 = x/y$ where $x \in Z$ and $y \in Z^+$

Moving the denominator over, $y * \log_8 9 = x$

Using the log power rule, $\log_8 (9^y) = x$

By definition of logs, $8^y = 9^x$

We can simplify this further $(2^3)^y = (3^2)^x \rightarrow 2^{3y} = 3^{2x}$

Considering that both x and y are integers, 2^{3y} and 3^{2yx} must also be integers. The fundamental theorem of arithmetic states that any integer can be represented as a unique product of prime numbers. The two exponentiations clearly don't share the same prime factors so walla! A clear contradiction. Because the assumption led to a contradiction, it must be false and $\log_8 9 \notin Q$.

How do we show that $\log_8 4$ is irrational?

Hm is $\log_8 4$ even irrational?

Proof. $\log_8 4 \notin Q$

We proceed by contradiction.

Assume that $\log_8 4 \in Q$. Reminder that $Q = \{ x/y \mid x \in Z \text{ and } y \in Z^+\}$

So, $\log_8 4 = x/y$ where $x \in Z$ and $y \in Z^+$

Moving the denominator over, $y*\log_8 4 = x$

Using the log power rule, $\log_8 (4^y) = x$

By definition of logs, $4^y = 8^x$

We can simplify this further $(2^2)^y = (2^3)^x \rightarrow 2^{2y} = 2^{3x}$

Considering that both x and y are integers, 2^{2y} and 2^{3x} must also be integers. The fundamental theorem of arithmetic states that any integer can be represented as a unique product of prime numbers. Unlike the two previous cases, the two exponentiations share prime factors. In fact, the two have the same factorization when $x = 2y/3$ and $y=3n$ where $n \in Z^+$. A contradiction is not reached…

Log Identity Practice

Question 34 (see above)

$$\log_{\sqrt{3}}(5) = \log_3 \Bigg(\qquad\qquad\qquad \Bigg)$$

Question 35 (see above)

$$\log_a(b)\log_a \Bigg(\qquad\qquad\qquad \Bigg) = 1$$

Question 38 (see above)

$$3^{\log_5(7)} = 7^{\log_{\square}(\square)}$$

Log Identity Practice

Question 34 (see above)

$$\log_{\sqrt{3}}(5) = \log_3 \Big(\qquad \Big)$$

$\log_{sqrt(3)}5 = \log_3 5 / \log_3(\sqrt{3})$

$= \log_3 5 / 0.5$

$= 2*\log_3 5$

$= \log_3 25$

Question 35 (see above)

$$\log_a(b)\log_a \Big(\qquad \Big) = 1$$

$\log_a(b)*\log_a(?) = 1$

$\log_a(?) = 1/ \log_a(b)$

$? = a^{1/log\_a(b)}$

$? = a^{log\_a(a)/log\_a(b)}$

$? = a^{log\_b(a)}$

Question 38 (see above)

$$3^{\log_5(7)} = 7^{\log_{\square}(\square)}$$

$\log_7(3^{log\_5(7)}) = \log_x(y)$

$\log_5(7)*\log_7(3) = \log_x(y)$

$(\log_7(7)/\log_7(5))*\log_7(3) = \log_x(y)$

$\log_7(3)/\log_7(5) = \log_x(y)$

$\log_5(3) = \log_x(y)$

$x=5,\ y=3$

How do we prove that $\forall n \in Z^{+}.\ \log_{n+2}(n+1) \notin Z$?

$\forall n \in Z^+.\ \log_{n+2}(n+1) \notin Z$ (Practice Problem 51)

Proof. $\forall\ n \in Z^+.\log_{n+2}(n+1) \notin Z$

We proceed by contradiction.

Assume that for all $n \in Z^+$, $\log_{n+2}(n+1) \in Z$.

- So, that means that for all $n \in Z^+$ there must exist an $x \in Z$ such that $\log_{n+2}(n+1) = x$
- By definition, $(n+2)^x = n+1$
- Because x is an integer, both sides are integers and they must have the same prime factorization as per the fundamental theorem of arithmetic.
    - Remember, all factors of $(n+2)^x$ are also that of (n+2)
    - So, all prime factors of (n+2) must also be that of (n+1).
- What next?

How do we prove that n+2 and n+1 can’t share prime factors?

$\forall n \in Z^+.\ \log_{n+2}(n+1) \notin Z$ Continued (Practice Problem 51)

To prove that for all $n \in Z^+$, n+2 and n+1 do not share prime factors, we proceed by contradiction

- We assume that there exists a prime number p which is a factor of n+2 and n+1.
    - So, $(n+2)/p \in Z^+$ and $(n+1)/p \in Z^+$
    - That means for some integer k, n+1=p*k
    - If we add one to both sides, n+2=p*k+1
    - Substituting that back into $(n+2)/p \in Z^+$, $(p*k+1)/p \in Z^+$
    - Simplifying that slightly, $(p*k)/p + 1/p \in Z^+$
    - Again, $k+1/p \in Z^+$
- Considering that p is a prime, which is greater than 1, we know that 1/p can not be an integer. With k being an integer, we know that k+1/p can’t be an integer and p is not a factor of n+2.
- This leads to a contradiction. n+2 and n+1 can’t share factors.

$\forall n \in Z^+ . \log_{n+2}(n+1) \notin Z$ Continued (Practice Problem 51)

Proof. $\forall\ n \in Z^+.\log_{n+2}(n+1) \notin Z$

We proceed by contradiction.

Assume that for all $n \in Z^+$, $\log_{n+2}(n+1) \in Z$.

- So, that means that for all $n \in Z^+$ there must exist an $x \in Z$ such that $\log_{n+2}(n+1) = x$
- By definition, $(n+2)^x = n+1$
- Because x is an integer, both sides are integers and they must have the same prime factorization as per the fundamental theorem of arithmetic.
    - Remember, all factors of $(n+2)^x$ are also that of (n+2)
    - So, all prime factors of (n+2) must also be that of (n+1).
- As in the previous slide, n+1 and n+2 share no prime factors and can't have the same prime factorization.
- Aha! Contradiction!

Because assuming that $\log_{n+2}(n+1)$ was an integer led to a contradiction, it must not be an integer.

It follows that $\forall\ n \in Z^+.\log_{n+2}(n+1) \notin Z$.

Questions?